ON THE RIEMANN FUNCTION AND IRREGULAR SINGULAR POINTS FOR AXISYMMETRIC BLACK HOLE COLLISIONS AT THE SPEED OF LIGHT

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Abstract. The news function providing some relevant information about angular distribution of gravitational radiation in axisymmetric black hole collisions at the speed of light had been evaluated in the literature by perturbation methods, after inverting second-order hyperbolic operators with variable coefficients in two independent variables. More recent work has related the appropriate Green function to the Riemann function for such a class of hyperbolic operators in two variables. The present paper obtains an improvement in the evaluation of the coefficients occurring in the second-order equation obeyed by the Riemann function, which might prove useful for numerical purposes. Eventually, we find under which conditions the original Green-function calculation reduces to finding solutions of an inhomogeneous second-order ordinary differential equation with a non-regular singular point.

The non-linear Einstein equations [1] ruling the gravitational field are so complicated that no exact solution has been obtained so far without making a number of simplifying assumptions. On the other hand, in the physically more relevant case of isolated gravitating systems, which are time-dependent, no simplifying assumption can be made apart from axisymmetry. Within this framework, many efforts have been devoted to analytic and numerical investigations of gravitational radiation produced in axisymmetric black hole collisions at the speed of light [2-5], since such events (although unlikely) are expected to lead to the largest amount of gravitational radiation ever studied at theoretical level.

The angular distribution of gravitational radiation is described in part by the news function [6], and the work by D'Eath and Payne [2-5] has shown that, in the above events, such a function can be obtained at second order in perturbation theory provided that one is able to solve a set of inhomogeneous hyperbolic equations taking eventually the form (m and n being integers)

$$\mathcal{L}_{m,n}\chi(q,r) = H(q,r). \tag{1}$$

With the notation in Refs. [2–5], q and r are the independent variables, the function H is related to the source term in the original set of equations, and the operator $\mathcal{L}_{m,n}$ reads

$$\begin{aligned}\mathcal{L}_{m,n} &= -(2\sqrt{2}+32q)\frac{\partial^2}{\partial q \partial r} + 4q^2\frac{\partial^2}{\partial q^2} + 64\frac{\partial^2}{\partial r^2} + 4(n+1)q\frac{\partial}{\partial q} \\ &- 16n\frac{\partial}{\partial r} + n^2 - m^2.\end{aligned} \tag{2}$$

The inverse of the operator $\mathcal{L}_{m,n}$ is an integral operator whose kernel is equal to the Green function $G_{m,n}(q,r;q_0,r_0)$ of Eq. (1).

The work in Ref. [7] has however pointed out that, after reduction of Eq. (1) to canonical form through the introduction of suitable new variables:

$$t \equiv \sqrt{1+16q\sqrt{2}}, \tag{3}$$

$$x \equiv r + \log\left(\frac{t-1}{2}\right) - \frac{8}{(t-1)} - 4, \tag{4}$$

$$y \equiv r + \log\left(\frac{t+1}{2}\right) + \frac{8}{(t+1)} - 4, \tag{5}$$

Eq. (1) can be solved for χ with the help of a standard integral formula [8] which involves the Riemann function. This is a valuable tool in the theory of hyperbolic equations in two variables with variable coefficients, but unfortunately it has not been much exploited (to our knowledge) in the literature on gravitational physics. Following Ref. [7], we consider the operator

$$L \equiv \frac{\partial^2}{\partial x \partial y} + a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} + c(x,y), \tag{6}$$

having defined w as the function of $x - y$ such that

$$w e^{\frac{w^2-1}{2w}} = e^{\frac{x-y}{8}}, \tag{7}$$

from which

$$t = \frac{1+w}{1-w}, \tag{8}$$

$$a(t) = a[t(w(x-y))] = \frac{1}{16}\frac{(1-t)(t+1)^2(2n+1+t)}{(t^4+4t^2-1)}, \tag{9}$$

$$b(t) = b[t(w(x-y))] = \frac{1}{16}\frac{(t+1)(t-1)^2(2n+1-t)}{(t^4+4t^2-1)}, \tag{10}$$

$$c(t) = c[t(w(x-y))] = \frac{(m^2-n^2)}{256}\frac{(t-1)^2(t+1)^2}{(t^4+4t^2-1)}. \tag{11}$$

The coefficient of the highest-order derivative in Eq. (6) is constant because

$$\mathcal{L}_{m,n}\chi = f L\chi = H,$$

where f is a function evaluated in Ref. [7], and hence we study the equation

$$L\chi = \frac{H}{f}$$

eventually, whenever f does not vanish. The (formal) adjoint of the operator L is then equal to

$$L^{\dagger} \equiv \frac{\partial^2}{\partial x \partial y} + A(x,y)\frac{\partial}{\partial x} + B(x,y)\frac{\partial}{\partial y} + C(x,y), \tag{12}$$

where

$$A \equiv -a, \tag{13}$$

$$B \equiv -b, \tag{14}$$

$$C \equiv c - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}. \tag{15}$$

Let now (ξ, η) be coordinates of a point P such that characteristics through it intersect a curve γ at points A and B, AP being a segment with constant y, and BP being a segment with constant x. As a function of x and y, the Riemann function R satisfies the homogeneous equation

$$L^{\dagger} R = 0, \tag{16}$$

the boundary conditions

$$\frac{\partial R}{\partial x} = bR \text{ on } AP, \quad \frac{\partial R}{\partial y} = aR \text{ on } BP, \tag{17}$$

and is equal to 1 at P. It is then possible to express the solution χ of Eq. (1) in the form [7]

$$\chi(P) = \frac{1}{2}[\chi(A)R(A) + \chi(B)R(B)] + \int_{AB} \left(\left[\frac{R}{2}\frac{\partial \chi}{\partial x} + \left(bR - \frac{1}{2}\frac{\partial R}{\partial x} \right) \chi \right] dx \right.$$
$$\left. - \left[\frac{R}{2}\frac{\partial \chi}{\partial y} + \left(aR - \frac{1}{2}\frac{\partial R}{\partial y} \right) \chi \right] dy \right)$$
$$+ \int\int_{\Omega} R(x,y;\xi,\eta)\frac{-H(x,y)}{256}\frac{(t-1)^2(t+1)^2}{(t^4+4t^2-1)}(x,y) dx\, dy, \tag{18}$$

Ω being a suitable bounded domain.

The main technical difficulty in Eq. (16) results from the derivatives occurring in the coefficient C; since $w(x-y)$ is found only implicitly from Eq. (7), and hence t, a, b, c

from Eqs. (8)–(11), it would be helpful not having to take numerical derivatives in (15) of a function which is only found numerically itself. The present note solves this specific problem by pointing out that, in Eq. (15), one has

$$\frac{\partial a}{\partial x} = \frac{\partial a}{\partial t}\frac{\partial t}{\partial w}\frac{\partial w}{\partial x}, \tag{19}$$

where, from Eq. (9),

$$\frac{\partial a}{\partial t} = -\frac{1}{16}\left[\frac{(4t^3 + 6(n+1)t^2 + 4nt - 2(n+1))}{(t^4 + 4t^2 - 1)} \right.$$
$$\left. + \frac{4t(1-t)(t+1)^2(t^2+2)(2n+1+t)}{(t^4 + 4t^2 - 1)^2}\right], \tag{20}$$

while Eq. (8) yields

$$\frac{\partial t}{\partial w} = \frac{2}{(1-w)^2}. \tag{21}$$

Moreover, from the logarithm of both sides of Eq. (7) we find

$$\log(w) + \frac{w}{2} - \frac{1}{2w} = \frac{(x-y)}{8}. \tag{22}$$

The derivative with respect to x of both sides of Eq. (22) yields eventually

$$\frac{\partial w}{\partial x} = \frac{1}{4}\left(1 + \frac{1}{w}\right)^{-2}. \tag{23}$$

Equations (19)–(23) yield a complete algorithm for the evaluation of the first derivative in C, i.e.

$$\frac{\partial a}{\partial x} = \frac{1}{2}\frac{\partial a}{\partial t}\frac{w^2}{(1-w^2)^2}, \tag{24}$$

where Eqs. (7), (8) and (20) should be inserted for the purpose of numerical analysis. An entirely analogous procedure holds for $\frac{\partial b}{\partial y}$ in Eq. (15), i.e.

$$\frac{\partial b}{\partial y} = \frac{\partial b}{\partial t}\frac{\partial t}{\partial w}\frac{\partial w}{\partial y}, \tag{25}$$

bearing in mind Eq. (10), and that

$$\frac{\partial w}{\partial y} = -\frac{\partial w}{\partial x}, \tag{26}$$

which leads to

$$\frac{\partial b}{\partial y} = -\frac{1}{2}\frac{\partial b}{\partial t}\frac{w^2}{(1-w^2)^2}, \tag{27}$$

with

$$\frac{\partial b}{\partial t} = -\frac{1}{16}\left[\frac{(4t^3 - 6(n+1)t^2 + 4nt + 2(n+1))}{(t^4 + 4t^2 - 1)}\right.$$
$$\left. + \frac{4t(t+1)(t-1)^2(t^2+2)(2n+1-t)}{(t^4+4t^2-1)^2}\right]. \tag{28}$$

Eventually, the coefficients A, B and C in the operator $L^\dagger$ are all expressed as functions of $w = w(x-y)$ with the help of previous equations as follows:

$$A = \frac{w(1+n-nw)}{4(1+2w-2w^2+2w^3+w^4)}, \tag{29}$$

$$B = \frac{w^2(n(-1+w)+w)}{4(1+2w-2w^2+2w^3+w^4)}, \tag{30}$$

$$64(1+3w+3w^4+w^5)^2 C = w^2\Big[(m^2-n^2)(1+4w+3w^2+3w^4+4w^5+w^6)$$
$$+ 4(n+1)(1-w^2-8w^3-w^4+w^6)\Big], \tag{31}$$

where we have used Eqs. (8)–(11), (13)–(15), (20), (24), (27) and (28). We have solved numerically Eq. (7) after re-expressing it in the form

$$8\log(w) + 4\left(w - \frac{1}{w}\right) = x - y, \tag{32}$$

for $x-y$ in the closed interval $[0.8, 10]$, at steps of 0.1. The variable w is then monotonically increasing and ranges from 1.05 through 1.83. The corresponding values of A, B and C

have been obtained upon insertion of w into Eqs. (29)–(31). For example, at the initial point $x - y = 0.8$, the coefficients A, B and C read

$$A = 0.059 - 0.0030n, \tag{33}$$

$$B = 0.065 + 0.0031n, \tag{34}$$

$$C = -0.0077 + 0.0038(m^2 - n^2) - 0.0077n, \tag{35}$$

respectively.

Work is now in progress in applying such formulae to the numerical evaluation of the Riemann function itself [9]. Hopefully, this will lead to more powerful tools for the investigation of gravitational radiation in the few cases where it is expected to be very rich, i.e. axisymmetric black-hole collisions at the speed of light [2–5].

Meanwhile, we are also making progress on alternative approaches to finding solutions of the equation $L\chi = \frac{H}{f} = \widetilde{H}$ that we started with. To exploit both Fourier transform theory and the fact that the coefficients in the operator L depend on t and hence, eventually, only on the difference $x-y$, we give up the canonical form of the operator L by introducing the independent variables

$$X \equiv \frac{x+y}{2}, \quad Y \equiv \frac{x-y}{2}. \tag{36}$$

Our inhomogeneous hyperbolic equation is then turned into

$$\widetilde{L}\chi(X,Y) = 4\widetilde{H}(X,Y) \equiv h(X,Y), \tag{37}$$

where

$$\widetilde{L} \equiv \frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2} + 2(A+B)\frac{\partial}{\partial X} + 2(A-B)\frac{\partial}{\partial Y} + 4C, \tag{38}$$

with coefficients depending only on Y. It is therefore more convenient to consider the Fourier transform of χ with respect to X (assuming that it exists), i.e.

$$\widehat{\chi}(k,Y) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi(X,Y) e^{-ikX} dX, \tag{39}$$

with inversion formula

$$\chi(X,Y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\chi}(k,X) e^{ikX} dk. \tag{40}$$

If also the right-hand side of Eq. (37) admits Fourier transform $\widehat{h}(k,Y)$ with respect to X, we therefore obtain the second-order equation

$$\left[\frac{\partial^2}{\partial Y^2} + 2(B-A)\frac{\partial}{\partial Y} + (k^2 - 4C - 2ik(A+B)) \right] \widehat{\chi}(k,Y) = -\widehat{h}(k,Y). \tag{41}$$

By virtue of Eqs. (9), (10), (13) and (14) we find

$$B - A = \frac{1}{8} \frac{(n+2)t(1-t^2)}{(t^4+4t^2-1)}, \tag{42}$$

$$B + A = \frac{1}{8} \frac{(t^2-1)(t^2+2n+1)}{(t^4+4t^2-1)}, \tag{43}$$

and we re-express Eq. (32) in the form

$$4\log(w) + 2\left(w - \frac{1}{w}\right) = Y. \tag{44}$$

Note now that, for each fixed value of k, Eq. (41) may be viewed as an inhomogeneous second-order ordinary differential equation for $\widehat{\chi}$. On denoting by M the operator in square brackets on the left-hand side of Eq. (41), and by χ_1 and χ_2 two linearly independent integrals of the homogeneous equation $M\widehat{\chi} = 0$, one can use the method of variation of parameters, according to which the general solution of Eq. (41) reads

$$\widehat{\chi} = \lambda_1 \chi_1(Y) + \lambda_2 \chi_2(Y) + v_1(Y)\chi_1(Y) + v_2(Y)\chi_2(Y), \tag{45}$$

with λ_1 and λ_2 constants, while v_1 and v_2 are chosen in such a way that

$$v_1' \chi_1 + v_2' \chi_2 = 0, \tag{46}$$

$$v_1' \chi_1' + v_2' \chi_2' = -\widehat{h} \equiv j(Y), \tag{47}$$

the prime denoting derivative with respect to Y. The system expressed by Eqs. (46) and (47) is solved by

$$v_1(Y) = -\int \frac{\chi_2(Y)j(Y)}{W(Y)} dY, \tag{48}$$

$$v_2(Y) = \int \frac{\chi_1(Y)j(Y)}{W(Y)} dY, \tag{49}$$

where W is the standard notation for the Wronskian of χ_1 and χ_2, i.e.

$$W(Y) \equiv \chi_1(Y)\chi_2'(Y) - \chi_2(Y)\chi_1'(Y). \tag{50}$$

In the course of investigating the singular points of Eq. (41), the technical difficulty results from the fact that its coefficients are ratios of polynomials in the t or w variable, but not in the Y variable itself by virtue of Eq. (44). A way out is obtained by turning Eq. (41) into an ordinary differential equation with w taken as the independent variable. For this purpose, we exploit Eq. (44) to find

$$\frac{dw}{dY} = \frac{1}{2}\left(1 + \frac{1}{w}\right)^{-2}, \tag{51}$$

and hence

$$\left[\frac{d^2}{dw^2} + g_1(w)\frac{d}{dw} + g_2(w)\right]\widehat{\chi}(k, w) = \sigma(w), \tag{52}$$

where

$$\begin{aligned} g_1(w) &\equiv \frac{2}{w^2}\left(1 + \frac{1}{w}\right)^{-1} + 4(B - A)\left(1 + \frac{1}{w}\right)^2 \\ &= \frac{2}{w(1+w)} - \frac{(n+2)}{2}\frac{(1+w)^2(1-w^2)}{w(1+(1+\sqrt{5})w+w^2)(1+(1-\sqrt{5})w+w^2)}, \end{aligned} \tag{53}$$

$$\begin{aligned} g_2(w) &\equiv 4\left(1 + \frac{1}{w}\right)^4 (k^2 - 4C - 2ik(A+B)) \\ &= 4\Bigg[\frac{(1+w)^4}{w^4}k^2 - \frac{1}{16w^2}\Bigg((m^2 - n^2)\frac{(1+w)^4}{(1+(1+\sqrt{5})w+w^2)(1+(1-\sqrt{5})w+w^2)} \\ &+4(n+1)\frac{(1-w^2-8w^3-w^4+w^6)(1+w)^2}{(1+(1+\sqrt{5})w+w^2)^2(1+(1-\sqrt{5})w+w^2)^2}\Bigg) \\ &- \frac{ik}{2}\frac{((n+1)(1+w^2)-2nw)(1+w)^4}{w^3(1+(1+\sqrt{5})w+w^2)(1+(1-\sqrt{5})w+w^2)}\Bigg], \end{aligned} \tag{54}$$

$$\sigma(w) \equiv 4\left(1+\frac{1}{w}\right)^4 j(Y(w)). \qquad (55)$$

Equation (52) has therefore a non-regular singular point at $w=0$, in that the coefficient $g_2(w)$ has a fourth-order pole therein. This corresponds to the point $x=-\infty$ in the original variable defined in Eq. (4). Moreover, the polynomial $p(w) \equiv 1+(1+\sqrt{5})w+w^2$ has real roots equal to

$$w_{1,2} \equiv -\frac{1}{2}(1+\sqrt{5}) \pm \sqrt{\frac{1}{2}(1+\sqrt{5})}. \qquad (56)$$

At these points, $g_1(w)$ has a first-order pole, and $g_2(w)$ has a second-order pole. The last pole occurs for $g_1(w)$ at $w=-1$, and it is of first order. No further poles occur at finite real values of w, since the polynomial $q(w) \equiv 1+(1-\sqrt{5})w+w^2$ has the complex conjugate roots

$$\frac{1}{2}(\sqrt{5}-1) \pm i\sqrt{\frac{1}{2}(\sqrt{5}-1)}.$$

By virtue of (54), Eq. (52) does not possess *normal integrals*. These are meant to be integrals admitting the factorization $e^{\Omega}u$ [10], where Ω is a polynomial in $\frac{1}{w}$ and u solves an equation with a Fuchsian singularity at the origin. The point $w=0$ would be therefore an essential singularity through the occurrence of the factor e^{Ω}. In our case, on re-expressing g_1 and g_2 in the form

$$g_1(w) = \frac{2}{w(1+w)} + \frac{F_1}{w},$$

$$g_2(w) = 4\frac{(1+w)^4}{w^4}k^2 + \frac{F_2}{w^2} + \frac{F_3}{w^3},$$

with obvious meaning of the functions F_1, F_2, F_3 by comparison with (53) and (54), the ansatz for normal integrals yields the following second-order equation for u:

$$\left[\frac{d^2}{dw^2} + (2\Omega' + g_1)\frac{d}{dw} + (\Omega'' + \Omega'^2 + g_1\Omega' + g_2)\right] u = e^{-\Omega}\sigma. \qquad (57)$$

Since u should obey an equation with regular singular point at the origin, we try to choose Ω in such a way that the coefficients of w^{-4} and w^{-3} vanish in Eq. (56). The former task is accomplished by choosing

$$\Omega' = 2ik\left(1 + \frac{1}{w}\right)^2,$$

which yields

$$\Omega'' + \frac{\Omega'}{w}\left(F_1 + \frac{2}{(1+w)}\right) + \frac{F_3}{w^3} = \frac{1}{w^3}\left(F_3 + 2ik(1+w)^2 F_1 + 4ikw\right) + \mathrm{O}(w^{-2}), \qquad (58)$$

where the term in round brackets multiplying w^{-3} does not vanish. This is why normal integrals cannot be found in our problem.

We have therefore to look for $\widehat{\chi}$ in the most general form [10]

$$\widehat{\chi}(k,w) = w^{\gamma} \sum_{l=-\infty}^{\infty} a_l w^l, \quad w \in]0, w_1[, \qquad (59)$$

where γ accounts for the multi-valuedness of the solution (any integer part being absorbed in the Laurent expansion valid in the open interval $w \in]0, w_1[$). This expansion is now inserted into the homogeneous equation associated to Eq. (52), i.e.

$$\left[\frac{d^2}{dw^2} + g_1(w)\frac{d}{dw} + g_2(w)\right]\widehat{\chi}(k,w) = 0, \qquad (60)$$

since the knowledge of two linearly independent integrals of Eq. (60) is sufficient to solve Eq. (52) by exploiting the method of variation of parameters previously described. For our purposes we consider Laurent expansions of g_1 and g_2 in the form

$$g_1(w) = \sum_{s=-\infty}^{\infty} a_s^0 w^s = \frac{h_1(w)}{w}, \quad w \in]0, w_1[, \qquad (61)$$

$$g_2(w) = \sum_{s=-\infty}^{\infty} b_s^0 w^s = \frac{h_2(w)}{w^4}, \quad w \in]0, w_1[. \qquad (62)$$

We therefore get from Eq. (60) the condition

$$\sum_{p=-\infty}^{\infty} \left[\sum_{l=-\infty}^{\infty} J(\gamma; l, p, k) a_l \right] w^p = 0, \tag{63}$$

having defined

$$J(\gamma; l, p, k) \equiv (\gamma + l)(\gamma + l - 1)\delta_{l,p+2} + (\gamma + l)a^0_{p-l+1}(k) + b^0_{p-l}(k). \tag{64}$$

Equation (63) leads to the infinite system of equations

$$\sum_{l=-\infty}^{\infty} J(\gamma; l, p, k) a_l = 0, \tag{65}$$

for all $p = -\infty, ..., +\infty$, where the coefficients a^0_l and b^0_l occurring in J are all known. Indeed one finds (cf. (53) and (54))

$$g_1(w) = \frac{1 - \frac{n}{2} - 2w - nw^2 + \mathrm{O}(w^3)}{w}, \tag{66}$$

$$g_2(w) = \frac{(-1 + 4k^2 - n) + 2(1 + 8k^2 + n - ik(1 + n))w + \mathrm{O}(w^2)}{w^4}, \tag{67}$$

if $w \in]0, w_1[$. By virtue of (64), Eq. (65) may be cast in the form

$$a_p = \sum_{l=-\infty}^{\infty} G_\gamma(p, l) a_l, \tag{68}$$

having defined

$$G_\gamma(p, l) \equiv -\frac{1}{(\gamma + p)(\gamma + p - 1)} \Big[(\gamma + l)a^0_{p-l-1} + b^0_{p-l-2} \Big], \tag{69}$$

where the general form of the coefficients a^0_l and b^0_l is found to be, in agreement with (53) and (54),

$$a^0_l \equiv n\alpha_{1,l} + \alpha_{2,l}, \tag{70}$$

$$b_l^0 \equiv m^2\alpha_{3,l} + n^2\alpha_{4,l} + n\alpha_{5,l} + \alpha_{6,l}. \tag{71}$$

The numerical coefficients $\alpha_{k,l}$, for all $k = 1, ..., 6$, can be inferred from (66) and (67). For example, one finds

$$\alpha_{1,1} = -\frac{1}{2}, \ \ \alpha_{2,1} = 1, \ \ \alpha_{1,2} = -2, \ \ \alpha_{2,2} = -2, \ \ \alpha_{1,3} = 7, \ \ \alpha_{2,3} = 12.$$

The polydromy parameter γ is found, at least in principle, by requiring that the homogeneous linear system expressed by Eq. (68) should have non-trivial solutions. Such a problem is currently under investigation. Hopefully, an intriguing link between black hole physics and the theory of infinite determinants [10] will be found to emerge.

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